The Quantum Black Hole Specific Heat Is Positive

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We suggest in this Letter that the Bekenstein–Hawking black hole entropy accounts for the degrees of freedom which are excited at low temperatures only and hence it leads to the negative specific heat. Taking into account the physical degrees of freedom which are excited at high temperatures, the existence of which we postulate, we compute the total specific heat of the quantum black hole that appears to be positive. This is done in analogy to the Planck's treatment of the black body radiation problem. Other thermodynamic functions are computed as well. Our results and the success of the thermodynamic description of the quantum black hole suggest an underlying atomic (discrete) structure of gravitation. The basic properties of these *gravitational atoms* are found.

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The notion of entropy for black holes was introduced by Bekenstein a long time ago [1] and it has been extensively used since then by many authors [2]. The Bekenstein entropy was defined as

$$S = \frac{1}{4}A = 4\pi M_{\mathcal{C}}^2 \ , \tag{1}$$

where A is the black hole horizon area and $M_{\mathcal{C}}$ is its total irreducible mass–energy. This irreducible mass-energy is left invariant by the 'reversible transformations in black hole physics' [3]. This formula also looked strangely familiar and it was reminiscent of physical quantities like entropy or adiabatic invariants of Hamiltonian dynamics at the same time. This was clear to those who discovered these properties of black holes. Here we propose to call the irreducible mass $M_{\mathcal{C}}$ and $I_{\mathcal{C}} = 4\pi M_{\mathcal{C}}^2$ the Christodoulou mass and the Christodoulou adiabatic invariant, respectively [3].

It has been shown that the entropy (1) leads to the *negative* specific heat for $x = \frac{J}{E^2} < x_c = (2\sqrt{3}-3)^{\frac{1}{2}} = 0.68125004...$, where J is the hole's angular momentum and E is its rest mass-energy.

From now on we focus our attention on the classical non-rotating Schwarzschild black hole for which the specific heat is negative

$$c_{bh} = -8\pi E^2 < 0 \ . \tag{2}$$

This property of negative specific heat is frequently used to argue that the canonical ensemble fails for gravitating systems, as (2) implies negative variance $\sigma^2(\bar{E}) = (\overline{\Delta E^2})$ of the normal Gaussian probability distribution [4]. We shall show that this problem can be avoided.

Let us emphasize the following two important physical points. First, the applicability of thermodynamic notions, and the entropy (1) in particular, suggests the underlying well hidden atomic or discrete structure behind the black hole dynamics. This observation seems to be known, though not pursued consequently, to some authors [5,6,7,8,9]. This situation we consider analogous to that of the atomic theory of matter at the turn of this century. In fact, this idea has been advocated by 't Hooft [10] and by one of the present authors [11,12].

Second, the negative specific heat indicates that we are dealing with an open system and some of the physical degrees of freedom have been neglected. These are the basic physical observations behind this paper. Our aim is to find thermodynamic functions of the quantum black hole, and in particular, its total specific heat $c_{tot} > 0$, entropy, and the partition function Z. For simplicity we consider a non–rotating black hole only because it is quite straightforward to generalize our results to other more general black holes. We claim that the Bekenstein–Hawking entropy takes into account the low temperature physical degrees of freedom whereas the high temperature degrees of freedom are neglected. This results in the physically unacceptable negative total specific heat (2).

In fact, this situation reminds us of the Wien law for the black body radiation. This prompted us to ask the question: What does the high temperature regime of quantum black hole behavior look like? Clearly, our understanding of this regime was missing (an analog of the Rayleigh–Jeans regime of the black body radiation) before we had realized that the answer is already there, but it was somehow well hidden until only recently [12]. The sole fact that the concept of entropy was invoked in the context of black hole physics should be interpreted as saying that we should take the fundamental idea of *Atomic Hypothesis* [11,12] seriously.

Hence, our idea is: *(i)* to take into account the high temperature degrees of freedom, *(ii)* to find out a counterpart of the Planck's interpolation formula [13], and *(iii)* to compute the complete thermodynamic functions of the quantum black hole. Finally, we shall show that our results imply underlying discrete or atomic structure behind a gravitating object (the quantum black hole).

To this end, we shall follow the method of Planck and Einstein [13,14], and we shall use a simple dimensional arguments. In view of what was said above, the total specific heat can be expressed as a sum of the Bekenstein–Hawking specific heat (2) and the specific heat coming from the high temperature degrees of freedom, and it should be positive: $c_{tot} = c_{bh} + c_{nc} > 0$, where the subscript nc stands for *non-collapse*. The meaning of the *non-collapse hypothesis* will be explained in the final section of this Letter. We expect that the specific heat of the high temperature degrees of freedom, $c_{nc} = c_{nc}(T)$, is a slowly varying function of T. This is not unlike the elementary example of the *'Einstein crystal'*, by which we mean the single frequency crystal [4].

We start with the Einstein fluctuation–dissipation theorem [14] for energy fluctuations

$$\sigma^2_{tot}(E) = \overline{\Delta E^2_{tot}} = \left(-\frac{\partial^2 S_{tot}}{\partial E^2}\right)^{-1} = T^2 c_{tot} \qquad (3)$$

that is valid for any closed quantum atomistic system and leads to the Gaussian probability distribution

$$P(\Delta E) \sim \exp\left(-(\Delta E)^2/2\sigma^2\right) \ . \qquad (4)$$

Treating the low temperature energy fluctuations as one random Gaussian variable and the high temperature fluctuations as the second independent Gaussian variable we see that the total distribution must be Gaussian. In fact, it is the convolution of both distributions

$$P_{tot}(\Delta E) = \int_{-\infty}^{+\infty} dy\ P_{bh}(\Delta E - y)\ P_{nc}(y) \sim \sim \exp\left(-(\Delta E)^2/2\sigma^2_{tot}\right) \qquad (5)$$

and, as a consequence, we have

$$\sigma^2_{tot} = \sigma^2_{bh} + \sigma^2_{nc} \ , \qquad (6)$$

where for the Bekenstein–Hawking fluctuations we have formally

$$\sigma^2_{bh} = -\frac{b\mu^2}{8\pi} \ , \qquad (7)$$

where μ is the Planck mass, and b is a dimensionless constant [15].

Now, the fundamental question arises: what expression should we choose for the high energy, or rather, for the high temperature fluctuations σ^2_{nc}? In fact, we have two constants of the dimension of energy: the Planck energy μ, that has been already used in (7), and the total energy E of the black hole. No other dimensional constants can be ascribed to a non–rotating gravitating object (Schwarzschild black hole). This was also the reason why it was considered earlier by one of us [12,17]. In order to understand the motivation beyond this choice we have to recall [18] that with the introduction by Planck of his constant h all fundamental constants were put in place and we have received from Planck the system of natural units. What is then left for us if not a large integer N? The meaning of that large integer N will become clear later on [8,12,16,17].

Hence, we can assume the following hypothesis [19]

$$\sigma^2_{nc} = aE^2 \ , \qquad (8)$$

where a is a *dimensionless* constant and E is the black hole mass–energy. The numerical value of the constant a and its relation to an integer N mentioned above will be given later, together with the discussion of our result within the context of physical ideas presented first in [11,12,17]. Our complete formula for the energy fluctuations now reads

$$\sigma^2_{tot} = -\frac{b\mu^2}{8\pi} + aE^2 \equiv a(E^2 - E_0^2) \ , \qquad (9)$$

where the constant E_0 is defined as $E_0^2 = \frac{b\mu^2}{8\pi a}$. The equation (9) is the basic formula we shall use to derive other thermodynamic functions.

Thermodynamic functions of the quantum black hole. It is a matter of simple integration to obtain from equations (3) and (9), after imposing the proper boundary condition, $T \to \infty$ when $E \to \infty$:

$$\frac{\partial S}{\partial E} = \beta \equiv \frac{1}{T} = \frac{1}{2aE_0} \ln\frac{E+E_0}{E-E_0} \ . \qquad (10)$$

This implies the temperature dependence of the mean energy

$$E(\beta) = E_0 \coth(aE_0\beta) \ , \qquad (11)$$

which we immediately recognize as the expression known from textbooks [4]. The low temperature asymptotics, $T \to 0$,

$$E \sim E_0 + 2E_0\ e^{-2a\beta E_0} \to E_0 \ . \qquad (12)$$

also explains the meaning of E_0. Thus, the constant E_0 should be interpreted as the minimal energy of the whole system, *i.e.* the total *zero point energy* of the quantum black hole. The total energy $E(T)$ is bounded from below by E_0, $E(T) \geq E_0$. For the high temperature regime, as for the Planck distribution, we have the following asymptotics, $T \to \infty$,

$$E \sim \frac{1}{a}\ T\ +\ \frac{1}{3}aE_0^2\ \frac{1}{T} \to \infty \ . \qquad (13)$$

The leading term in this formula is clearly the Rayleigh-Jeans result and it shall be interpreted as the energy equipartition rule. Now, using the standard formula for the canonical ensemble: $E = -\partial/\partial\beta \ln Z[\beta]$ one has the partition function in the following simple form

$$Z[\beta] = \left(\frac{1}{2\sinh\frac{\beta\epsilon}{2}}\right)^{1/a} \quad , \tag{14}$$

where $\epsilon = 2aE_0$ has been defined as

$$\epsilon = \mu\,\sqrt{\frac{ab}{2\pi}}\ . \tag{15}$$

Integrating (10) once again we obtain the total entropy as a function of energy

$$S(E) = S_0 + \frac{1}{2a}\frac{E}{E_0}\ln\frac{E+E_0}{E-E_0} + \frac{1}{2a}\ln\frac{E^2-E_0^2}{E_0^2}\ , \tag{16}$$

where S_0 is an integration constant independent of E and a. It is convenient to set it to zero. This point will be discussed shortly later. Eq. (16) gives the following asymptotics

$$\begin{aligned} S(E) &\sim \frac{1}{a}\ \ln\frac{E}{E_0} \to \infty \quad (E\to\infty)\ ,\\ S(E) &\to \frac{1}{a}\ln 2 \quad (E \to E_0)\ . \end{aligned} \tag{17}$$

The entropy S can be computed as a function of temperature T, $\beta = 1/T$, directly from the partition function (14). Up to a constant S_0 mentioned earlier this gives

$$S(\beta) = -\frac{1}{a}\ \ln\sinh\frac{\beta\epsilon}{2} + \frac{\beta\epsilon}{2a}\ \coth\frac{\beta\epsilon}{2}\ . \tag{18}$$

Finally, from (10) we calculate the total specific heat

$$c_{tot} = \frac{a\ E_0^2\ \beta^2}{\sinh^2 aE_0\beta}\ . \tag{19}$$

c_{tot} is always positive and has the following asymptotics

$$\begin{aligned} c_{tot}(T) &\sim 4aE_0^2\ \frac{1}{T^2}\ e^{-2aE_0/T} \to 0 \quad (T\to 0)\ ,\\ c_{tot}(T) &\to \frac{1}{a} - \frac{1}{3}\frac{a\ E_0^2}{T^2} \quad (T\to\infty)\ . \end{aligned} \tag{20}$$

The gravitational quanta. In order to discuss the physical consequences of the results from the previous section, we have to determine first what is the physical meaning of the dimensionless constant a. Our choice is

$$a = \frac{1}{N}\ , \qquad N - \text{integer}\ . \tag{21}$$

There are several arguments in favor of this choice. From the point of view of this Letter the most important fact is that the statistical sum (14) can be factorized as a product of N independent one–particle partition functions Z_1. Now, we can rewrite (14) as

$$Z = Z_1^N\ , \qquad Z_1 \equiv \frac{1}{2\sinh\frac{\beta\epsilon}{2}}\ . \tag{22}$$

The choice of the integer value for N is forced on us by the probabilistic interpretation. An integer N is also quite appealing on aesthetical grounds. Whenever an integer appears naturally from some simple hypothesis we should always be on the lookout for something fundamental [13,18,14]. Finally, eq. (21) is in agreement with the *new gravitational noncommutative mechanics* introduced in [11,12,17,16], where the wider spectrum of arguments has been given. The basic ideas of the *new gravitational mechanics* [11] were discussed also in the context of *gedanken experiments* [22] before [11] was published. The *non-collapse hypothesis*, which was implemented by *the postulate of the noncommutative Manin torus replacing the uniformizing complex torus in the GRT Kepler problem upon transition to the new gravitational mechanics* [11], was tested with the thought experiments using the Planckian energy gedanken accelerators [22].

The constants E_0 (the total energy of the black hole in the zero temperature limit) and ϵ can be expressed as

$$E_0 \ =\ N\ \frac{\epsilon}{2}\ =\ \sqrt{N}\ \mu\ \sqrt{\frac{b}{8\pi}}\ , \tag{23}$$

$$\epsilon \ =\ \frac{1}{\sqrt{N}}\ \mu\ \sqrt{\frac{b}{2\pi}}\ . \tag{24}$$

One can see immediately from (22) that Z_1 is the partition function of a quantum harmonic oscillator

$$Z_1 \ \equiv\ \frac{e^{-\beta\epsilon/2}}{1-e^{-\beta\epsilon}} \ =\ \sum_{n=0}^{\infty}\ e^{-\beta E_n}\ , \tag{25}$$

with the energy levels E_n defined as

$$E_n \ =\ \left(n + \frac{1}{2}\right)\ \epsilon\ . \tag{26}$$

Hence, our gravitational quanta are bosons: they do obey the Bose–Einstein statistics and eq. (13) states that at high temperatures we have the equipartition of energy in the leading order. Obviously, ϵ is the harmonic oscillator level spacing which depends universally on N [12].

At this point we can go back to the formula (16) for the entropy and, in particular, to its zero temperature limit (17). We can see now that the choice of the constant of integration, $S_0 = 0$, was quite fortunate. Indeed, the remaining term with the overall factor of N

$$S(0) \ =\ N\ln 2\ , \tag{27}$$

can be easily connected to the ground state degeneracy $d_N = e^{S(0)}$. From (27) we have $d_N = 2^N$. This is quite a remarkable formula as it implies that each elementary gravitational quantum can be in *two* fundamental states [20]. We simply encounter here a Z_2 quantum number. An analogy to the spin variables invites itself quite naturally. However, we *do not* suggest here at this point that this double–valuedness is related to the usual notion of

spin of ‘elementary particles’ as our *gravitational atoms* are much more fundamental than the so-called ‘elementary particles’ and such a proposal would be presumably too far fetched. The *gravitational atoms* seem to be a part of the physical objective reality [12] which should find its confirmation in observations and experimental data. One way of finding out how real they are is to propose an experiment in which their existence will be tested indirectly, in the same way as the theory of Brownian motion [4] due to Einstein, and Smoluchowski, has led to the experimental confirmation of the existence of atoms and molecules. The *gravitational atoms* do exist on a deeper level of the physical reality. The natural scale for gravitational atoms is the Planck scale.

We can recognize now, that the formula for E_0 and (21) imply the mass–energy quantization of the type derived first in [8]: $E^2(0) = M_0^2 = P_\mu P^\mu = m^2 N$, with some Planckian mass scale m. These considerations suggest to us that the invariant mass $M_0^2 = P_\mu P^\mu$ is quantized and it can be viewed as consisting of N Planckian mass scale *gravitational atoms* (see eq. (23)): $M_0^2 = N\, m^2$, where $m^2 = \frac{b\mu^2}{8\pi}$.

One can interpret this last formula as saying that at the zero temperature the quantum system which we call here the quantum black hole behaves as N free identical gravitational atoms of mass m. We can see that it is the invariant mass squared which is additive at the zero temperature. It is perhaps better to say that *gravitational quanta* are collective excitations in the system of N *gravitational atoms*. From (24) and (25) it is also clear that the large N limit ($N \to \infty$) or, equivalently, $1/N \to 0$ corresponds to the classical limit. In this limit we get a massive gravitating object and Einstein's general relativity is recovered. This should be understood properly, as the *classical limit* $\frac{G}{c^2} = K \to 0$ is highly nontrivial. This is similar to the transition from quantum mechanics to classical mechanics.

Finally, we would like to comment on the *non–collapse hypothesis* which was mentioned above. It is clear from the derivation of the fundamental results in this Letter that the negative specific heat and the gravitational collapse are intimately connected [12,16]. Also, the postulate of the missing degrees of freedom presented in this Letter implies directly that the unitarity requirement is broken in any quantum theory which does not take them into account. Hence the *S–matrix* postulate of ‘t Hooft [21] is similar if not equivalent to the *non–collapse hypothesis* first proposed in [11,12,17,16]. The *non–collapse hypothesis* implies that elastic channels for collision of ‘small black holes’ are present and therefore something akin to equipartition of energy is possible in some range of temperatures. The exact relationship between the *non–collapse hypothesis*, which is presented in this Letter in the context of the fluctuation–dissipation theorem, and the issue of unitarity is quite subtle and it is clearly beyond the scope of this Letter. However, in view of the reaction this point has received recently [16,17], we plan the more detailed pedagogical paper in the nearest future.

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